How much is convenient to defect ?
A method to estimate the cooperation probability in Prisoner's Dilemma and other games

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Sommario

In many cases the Nash equilibria are not predictive of the experimental players' behaviour. For some games of Game Theory it is proposed here a method to estimate the probabilities with which the different options will be actually chosen by the players. These probabilities can also be interpreted as competitive mixed strategies. The method is shaped on the Prisoner's Dilemma, then generalized for asymmetric tables, N players and N options. It is adapted to other conditions like Chicken Game, Battle of the Sexes, Stag Hunt and then applied to other games like Diner's Dilemma, Public Goods Game, Traveler's Dilemma and War of Attrition. These games are so analyzed in a probabilistic way that is consistent to what we could expect intuitively, overcoming some known paradoxes of the Game Theory.

1 Nash equilibria are not always predictive

In many cases the Nash equilibria are not predictive of the experimental players' behaviour.

For instance, "in the Public Goods Game repeatedly played, experimental observations show that individuals do not play the predicted noncooperative equilibria", at least not immediately (Ahn & Janseen, 2003, Adaptation vs. Anticipation in Public-Good Games).

"In the Traveler's Dilemma it seems very unlikely that any two individuals, no matter how rational they are and how certain they are about each other's rationality, each other's knowledge of each other's rationality, and so on, will play the Nash equilibrium" (Kaushik Basu, The Traveler's Dilemma: Paradoxes of Rationality in Game Theory; American Economic Review, Vol. 84, No. 2, pages 391-395; May 1994).

This paradox has led some to question the value of game theory in general, while others have suggested that a new kind of reasoning is required to understand how it can be quite rational ultimately to make non-rational choices. In this sense, Douglas Hofstadter proposed the theory of Superrationality: "it is assumed that the answer to a symmetric problem will be the same for all the superrational players. The strategy is found by maximizing the payoff to each player, assuming that they all use the same strategy. In the Prisoner's Dilemma two superrational players, each knowing that the other is also a superrational player, will cooperate" (Wikipedia, ref. Douglas R. Hofstadter, 1985, Metamagical Themas, Basic Books).

I will try to quantify this concept, trying to associate to each option a probability that estimates how many players will actually choose that option. This probability could suggest how much that option is convenient, depending on the given parameters of the game. These estimated probabilities could also be interpreted as competitive mixed strategies adopted by the players; given N options, players playing with mixed strategies will use a randomizing device, set to give the result "play option i" with probability p_i, where $i \in \mathbb{N}, 1 \leq i \leq N$. Only the competitive case and not the cooperative ones will be analyzed.

This study was actually born from a practical need, i.e. preparing some equilibrated bimatrixes (more simply we will call them "tables") for a game of mathematics and diplomacy, based on the Prisoner's Dilemma. We will speak about it after presenting the proposed model.

2 Prisoner's Dilemma, other games and Hofstadter's Superrationality

The Prisoner's Dilemma was originally framed by Merrill Flood and Melvin Dresher working at RAND Corporation in 1950. Albert W. Tucker formalized the game with prison sentence payoffs and gave it the prisoner's dilemma name (Poundstone, 1992). A classic example of the prisoner's dilemma is presented as follows.

Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated the prisoners, visit each of them to offer the same deal. If one testifies for the prosecution against the other (defects) and the other remains silent (cooperates), the defector goes free and the silent accomplice receives the full 10-year sentence. If both remain silent, both prisoners are sentenced to only six months in jail for a minor charge. If each betrays the other, each receives a five-year sentence. Each prisoner must choose to betray the other or to remain silent and each one is assured that the other would not know about the betrayal before the end of the investigation. Both care much more about their personal freedom than about the welfare of their accomplice. (Source: Wikipedia).

We will consider a table of Prisoner's Dilemma, where instead of jail years to minimize there are money prizes to maximize; given $a, b, c, d \in \mathbb{R}$, we define the table (a, b, c, d) in the following way: if both players cooperate, both receive b; if both defect, both receive c; if one defects and the other cooperates, the first receives a and the second d. Often these values are indicated with T, R, P, S, but in this document we prefer call them a, b, c, d for several reasons, among them the fact that we will deal with another p indicating a probability.

In the rest of the document we will analyze several conditions, sometimes they are studied in the literature as a specific game, with a specific name:

$$1.1) \quad a > b > c, d$$

In the Prisoner's Dilemma $c > d$:

$$1.2) \quad a > b > c > d$$

while in the Chicken Game $b > d \geq c$:

$$1.3) \quad a > b > d \geq c$$

In the Battle of the Sexes:

$$1.4) \quad a > d > c \geq b$$

In the Stag Hunt:

$$1.5) \quad b > a > c > d$$

We will also study an anomalous case, that we will call *the Translators* (see later):

$$1.6) \quad a > c \geq b > d$$

We will start from the following remark: in a table (a, b, c, d) like (100, 51, 50, 0) $b-c$ is so small comparing to $a-b$ and $c-d$ that each player will probably defect, according to the Nash equilibrium. Instead, in a table like (101, 100, 1, 0), like that one analyzed by Hofstadter for his Theory of Superrationality, the advantages of defection $a - b$ and $c - d$ are so small that is almost not worth the risk to come out with (1, 1) instead (100, 100), and so each player will tend more to cooperation than to defection.

Now, we will analyze the following situation. We have a large number of rational players that are going to play one time the Prisoner's Dilemma; they are divided in pairs and every pair plays with the same table (a, b, c, d); we will try to find a probability p that could estimate how many players will cooperate, consistently with the previous considerations. We set $q = 1-p$, as the defection probability. So $\forall a, b, c, d \in \mathbb{R}$ respecting the condition 1.2) we would like to find a $0 < p < 1$, reaching $p = 0$ only in the limit case $b \leq c$, and $p = 1$ only in the limit case: $a \leq b$ and $c \leq d$. If we suppose that the players are adopting mixed strategies with different cooperation probabilities p_i, this p will also represent the average of these p_i. This estimation should depend only on the given parameters (a, b, c, d), and not on the history of the game, that's why we are not going to consider iterated games, Fictitious Play or Evolutionary Stable Strategies.

3 Maximin criterion

4 Maximization of expected payoff

5 Maximization of expected payoff, given the opponent cooperation probability

6 The proposed estimation of the cooperation probability

Let us examine a fourth way. Under the condition 1.2) ($a > b > c > d$), we try to think the possible behaviour of a generical player X playing against a generical player Y. We can assume that the probability p_x that s/he will cooperate is proportional to $b-c$ (the benefit received by two cooperating players comparing with two defecting players), while the probability q_x that s/he will defect is proportional to $p_y(a-b)+q_y(c-d)$ (the benefits received by player X defecting instead of cooperating, weighted according to the cooperation and defection probabilities of player Y). We could start our reasoning giving to p_y an arbitrary initial value $0 \leq p_0 \leq 1$ ($q_0 = 1-p_0$). Said $b-c=\phi$ and $p_0(a-b)+q_0(c-d)=\chi_0$, the first estimation of p_x is $p_1=\phi/(\phi+\chi_0)$. Now, using this first estimation of p_x, we can try to give a first estimation of p_y, considering that, consistently with the previous reasoning, p_y is proportional to ϕ and q_y is proportional to $p_1(a-b)+q_1(c-d)=\chi_1$, so $p_2=\phi/(\phi+\chi_1)$. We can continue this procedure giving a second estimation of p_x, then a second estimation of p_y, and so on. Said $p(a-b)+q(c-d)=\chi$, this recursive sequence p_i, independently from the starting point p_0, will tend to: $p=\phi/(\phi+\chi)$.

From there we obtain a second degree equation:

$$3) \quad p^2(a-b-c+d)+p(b-d)+(c-b)=0$$

with solution

$$4.1) \quad p=\frac{d-b\pm\sqrt{(b-d)^2+4(b-c)(a-b-c+d)}}{2(a-b-c+d)}$$

with $a-b-c+d \neq 0$.

If $a-b-c+d=0$ from the 3) we have more simply

$$4.2) \quad p=\frac{b-c}{a-c}$$

For example, going back to what we have seen in paragraph 2, in the Hofstadter's table (101, 100, 1, 0) the 4.2) gives $p=99\%$, and in the table (100, 51, 50, 0) the 4.1) gives $p\approx 1.96\%$; these results are consistent with what we could expect intuitively.

7 The probability is univocal and always defined

8 Equiprobability condition

9 Prisoner's Dilemma with n players

10 Prisonner's Dilemma with asymmetric tables

11 Translators condition

We come back to the case for 2 players. We analyze some other conditions that we will find later in some applications and that are useful to define some boundaries of the proposed estimation.

We will examine first the condition 1.6) $a > c \geq b > d$, and we will call it "Translators".

12 Stag Hunt condition

An interesting case is the condition 1.5) $b > a \geq c > d$, called Stag Hunt. In "Discours sur l'origine de l'inégalité parmi les Hommes" (1754) Jean-Jacques Rousseau described a situation in which two individuals go out on a hunt: "Voilà comment les hommes purent insensiblement acquérir quelques idées grossières des engagements mutuels, et de l'avantage de les remplir mais seulement autant que pouvait l'exiger l'intérêt présent et sensible ; car la prévoyance n'était rien pour eux, et, loin de s'occuper d'un avenir éloigné, ils ne songeaient même pas au lendemain. S'agissait-il de prendre un cerf, chacun sentait bien qu'il devait pour cela garder fidèlement son poste ; mais si un lièvre venait à passer à la portée de l'un d'eux, il ne faut pas douter qu'il le poursuivit sans scrupule, et qu'ayant atteint sa proie il ne se soucia fort peu de faire manquer la leur à ses compagnons". So, if both collaborate in hunting the stag, they both receive b, if both will hunt the less worthy hare both receive c ($c < b$), if one hunts the hare while the other remains alone trying to hunt the stag, the first one receives a ($b > a \geq c$) and the second one receives d ($d < c$). Here we have $\phi = b - c + p(b - a)$ and $\chi = q(c - d)$. We could expect that under this condition should be always $p = 1$. The equation in this case can be expressed as:

$$10.1) \quad (p-1)(p - \frac{c-b}{-a+b-c+d}) = 0$$

with $-a+b-c+d \neq 0$, therefore actually one root is always 1.

Say the second root $r_2 = (c-b)/(-a+b-c+d)$. If $r_2 \geq 1$ the attractor of the recursive sequence for the set $[0,1]$ is 1, and so $p = 1$. If $0 \leq r_2 < 1$ we find

$$10.2) \quad 0 \leq \frac{b-c}{a-d} < 1/2$$

In this case r_2 is the attractor for the set $[0,1]$, and so $p = r_2$. This result could be unexpected, as $a < b$, but actually it is consistent with the problem:

for example, in the condition $a - d >> b - c$ (that respects the 10.2) the risk to receive d is not comparable to the small advantage $b - a$, so p is small. If $a - b + c - d = 0$, we find $\chi = q(b - a) \Rightarrow p = \frac{b-c}{b-c} = 1$, and it agrees with 10.2), being $\frac{b-c}{a-d} = 1 \Rightarrow r_2 \geq 1$.

13 Chicken condition

14 Battle of the Sexes condition

15 Application to a game based on the Prisoner's Dilemma

16 Application to the Diner's Dilemma

17 Application to the Public Goods Game

18 Application to the Traveler's Dilemma

This game was formulated in 1994 by Kaushik Basu and goes as follows. An airline loses two suitcases belonging to two different travelers. Both suitcases happen to be identical and contain identical antiques. An airline manager tasked to settle the claims of both travelers explains that the airline is liable for a maximum of \$100 per suitcase, and in order to determine an honest appraised value of the antiques the manager separates both travelers so they can't confer, and asks them to write down the amount of their value at no less than \$2 and no larger than \$100. He also tells them that if both write down the same number, he will treat that number as the true dollar value of both suitcases and reimburse both travelers that amount. However, if one writes down a smaller number than the other, this smaller number will be taken as the true dollar value, and both travelers will receive that amount along with a bonus/malus: \$2 extra will be paid to the traveler who wrote down the lower value and a \$2 deduction will be taken from the person who wrote down the higher amount. The challenge is: what strategy should both travelers follow to decide the value they should write down?

Say r the maximum value, s the minimum value, t the bonus, with $r > s \geq t > 0$ ($r, s, t \in \mathbb{R}$). The two players have $N + 1$ options: given $v = (r - s)/N$ they can play $s, s + v, s + 2v, ..., s + iv, ..., r$, with $i, N \in \mathbb{N}$.

We will try to apply again the considerations in paragraph 6 to the case with the 2 options $s + iv$ and $s + jv$ ($j \in \mathbb{N}$); said T_{ij} the table considering the two options i or j, we define p_{ij} as the cooperation probability in T_{ij}, so the probability to play the biggest value between $s + iv$ and $s + jv$. We obtain the following values, with $i > j$:

$a = s + jv + t$
$b = s + iv$
$c = s + jv$
$d = s + jv - t$.

We find $a > b \Rightarrow i - j < t/v$, and $t > 0 \Rightarrow c > d$.

$b > c \Rightarrow i > j$, already known.

Applying the 4.3, we obtain the cooperation probability p_{ij} (for $i > j$ it is the probability to play $s + iv$):

$$22) \quad p_{ij} = \frac{-(t + (i-j)v) + \sqrt{(t + (i-j)v)^2 - 4(i-j)^2v^2}}{2(j-i)v}$$

If $i - j \geq t/v \Leftrightarrow b > a$ we are under the condition $b > a > c > d$, so we must apply the 10.1).

If $i < j$ we just swap i and j, obtaining the same cooperation probability, that in this case will be the probability to play $s + jv$. We can see that p_{ij} depends on $\mid i - j \mid$, but not on i or j separately.

From the equation 6, we can see that the equiprobability condition is:

$$23) \quad 3(b - c) > (a - d) \Leftrightarrow i - j > 2t/3v$$

With the same method used for the Public Goods Game, we have $U_i = \sum_{j=0}^{i-1} p_{ij} + \sum_{j=i+1}^{N} q_{ij}$, $W = \sum_{i=0}^{N} U_i$ and $p_i = U_i/W$.

For a simple numerical example, let us set $r = 4$, $s = 2$, $t = 2$, $N = 2$ (3 options), $v = 1$. We can see that for $\mid i - j \mid= 1$ we have $p_{ij} \approx 38\%$; for $\mid i - j \mid\geq 2$, considering the 10.2), we can check that $(b-c)/(a-d) \geq 1/2$, hence we have always $p_{ij} = 1$. We obtain:

$p_0 = p(2) \approx 20.6\%$,

$p_1 = p(3) \approx 33.3\%$,

$p_2 = p(4) \approx 46.1\%$.

In the original problem, with $r = 100$, $s = 2$, $t = 2$, $N = 98$ (99 options), $v = 1$; here also for $\mid i - j \mid\geq 2$, $(b - c)/(a - d) \geq 1/2$, hence we have always $p_{ij} = 1$. We obtain

$p_{98} = p(100) \approx 2.01\%$

$p_0 = p(2) \approx 0.0128\%$

$p_i = p(i + 2) \approx i0.0206\%$.

These results are consistent with what we could expect intuitively.

In the article *The Traveler's Dilemma* (Basu, Kaushik. Scientific American Magazine; June 2007) experimental results are reported, where $r = 200$, $s = 80$, $N = 120$ (121 options), $v = 1$. For $t = 5$ the average amount proposed by the players was $\mu = 180$, and for $t = 80$ it was $\mu = 120$. Both results are quite far from the Nash equilibrium ($s = 80$). With our method we obtain: for $t = 5$, $\mu = \sum_{i=0}^{N}(s + iv)p_i \approx 160$ and for $t = 80$, $\mu \approx 144$. Our model is not too far from the experimental results.

19 Application to the War of Attrition

20 Conclusions

The proposed approach seems to describe quite well some classical games of the game theory, using an estimation of the players' behaviour to solve some paradoxes. This estimation can be seen as a convenience index for the different options. We can see these results also from the point of view of competitive mixed strategies: assuming that players are playing different mixed strategies, this estimation represents the average of these played strategies. It is possible to apply this approach to many other games, only some applications were showed here. Another interesting result could be to extend this method to calculate the probability density associated to a continuous range of options; for example, in the Public Goods Game, in the Traveler's Dilemma and in the War of Attrition, each player could choose whatever real number in a fixed range.

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